*Journal of Engineering Mathematics* **24**: 151–165, 1990.



# A free-boundary problem for viscous fluid flow in injection moulding

J.C.W. VAN VROONHOVEN and W.J.J. KUIJPERS*

*Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (*author for correspondence)*

Received 4 July 1989; accepted in revised form 11 December 1989

**Abstract.** The injection of a viscous fluid into a mould formed by two parallel plates is considered. The flow front is supposed to move at constant speed. It is assumed that there is complete adherence between the fluid and the mould walls, and that the environmental pressure is constant. For a Newtonian fluid the problem is described in terms of two analytic complex functions. The shape of the fluid surface is calculated by means of a conformal-mapping technique, which leads to a Hilbert problem. The results are compared with known finite-element simulations.

## 1. Introduction

Injection moulding is a process for the manufacture of products of a thermoplastic material. Before moulding this material is heated beyond its melting point and then injected under pressure into a mould cavity. In order to achieve complete filling of the cavity, pressure is maintained during the cooling stage of the process. When the material has solidified and has attained the mould shape, the product is ejected from the mould.

During the filling stage the fluid surface advances until the mould cavity has been filled completely. As the shape of the surface is unknown, it is called a free boundary. The flow directly behind the front resembles the flow of a fountain. Streamlines, initially parallel, diverge when approaching the flow front. Fluid particles decelerate and move outwards to the cavity walls. This characteristical flow is therefore referred to as "the fountain effect". We focus our attention on the free boundary, i.e. the fluid surface. Since it takes some time until the melting temperature has been reached, solid layers will form at a certain distance of the flow front. So the effects of cooling and solidification can be neglected and we may restrict ourselves to the isothermal problem.

A classical problem in injection moulding is the flow of a viscous fluid between two parallel plates. The solution of this problem also gives a realistic impression of the fountain flow and of the shape of the fluid surface in the case of a mould of a more general geometry. We use the model of an incompressible Newtonian fluid. This is a linear, homogeneous, isotropic fluid, for which the stresses depend linearly on the strain rate. The equations describing the fluid behaviour are the incompressibility condition, the equilibrium of the stresses and the constitutive equations. The boundary conditions for the typical geometry of the fountain flow are determined by the following assumptions. The fluid will fully adhere to the mould walls, which means that no slip will occur. It is supposed that the fluid surface advances at a constant speed and that its shape does not alter. The environmental pressure is assumed to be constant and the effects of surface tension are neglected. In order to complete the mathematical formulation of the problem we need the condition that the velocity field is fully developed far behind the flow front. This fully developed velocity field for a Newtonian fluid is the Poiseuille flow.

This free-boundary problem has been analysed by Mavridis, Hrymak and Vlachopoulos [7], who used a finite-element simulation, and by Dierieck [2], who introduced a stream function to satisfy the incompressibility condition. To determine the shape of the free boundary both authors use an iterative method involving much numerical effort. For a more specific treatment of the fountain effect we employ an alternative method of solution based on the theory of complex functions. The incompressibility condition is satisfied by the introduction of a stream function and the equations of equilibrium of the stresses are satisfied by the introduction of a stress function. The constitutive equations relate these two functions to each other. As a consequence these functions are solutions of the biharmonic equation. All equations are satisfied by the introduction of two independent analytic functions. These analytic functions are completely determined by the boundary conditions. The problem is solved by a conformal-mapping technique leading to a Hilbert problem. This method of solution is often used in the theory of linear elasticity, e.g. see England [3], Muskhelishvili [9], and can be applied to the current problem, because the behaviour of an incompressible Newtonian fluid and of an incompressible linear-elastic (Hookean) material is governed by essentially the same equations. The theory presented by Muskhelishvili and England is usually applied to elastic bodies of a shape that is known beforehand. In this paper we deal with a free-boundary problem, which means that the flow region has an unknown shape. Nevertheless, the conformal mapping technique can still be used to calculate the velocity field together with the shape of the free boundary. The theory of complex functions was also used by Garabedian [4], who derived the solution of several inverse problems. He prescribed the shape of the free boundary and calculated the velocity of the fluid for some specific geometries.

## 2. The fundamental equations

An incompressible Newtonian fluid is injected into the space between two parallel plates at mutual distance $2h$. The flow front moves with a constant velocity $V_f$ relative to the fixed walls as shown in Fig. 2.1.

The problem will be described in a moving frame of reference defined by

$$x = \frac{X - V_f t}{h}, \quad y = \frac{Y}{h}, \tag{2.1}$$

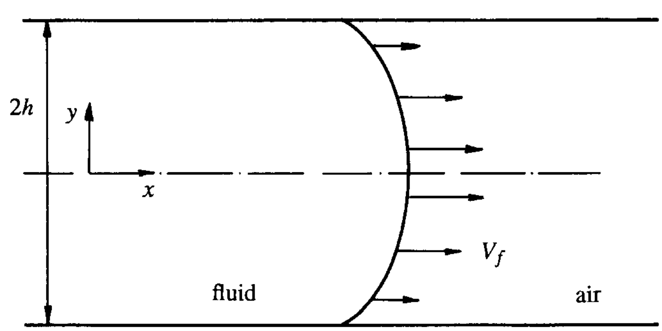


*Fig. 2.1.* The fixed frame of reference.

where $X$ and $Y$ are Cartesian coordinates in a fixed frame and $t$ denotes the time. All quantities describing the flow are functions of the coordinates $x$ and $y$ only.

Let A and C be the points where the fluid surface makes contact with the walls. The $x$-coordinate in the points A and C is chosen to be zero. The plane $y = 0$ corresponds to the plane of symmetry. Let B be the point of the surface where $y = 0$ (see Fig. 2.2). The $y$-coordinates of the planes AE and CD are equal to $-1$ and $+1$ respectively.

The dimensionless velocity of the fluid relative to the flow front is denoted by

$$\mathbf{v} = u(x, y)\mathbf{e}_x + v(x, y)\mathbf{e}_y \,. \tag{2.2}$$

The absolute velocity of the fluid relative to the fixed walls is then given by

$$\mathbf{V} = V_f(1 + u(x, y))\mathbf{e}_x + V_f v(x, y)\mathbf{e}_y \,. \tag{2.3}$$

The stress tensor in the point $(x, y)$ is denoted by $T$. The dimensionless stress tensor $\tau$ in $(x, y)$ is defined by the relation

$$T = \frac{2\eta V_f}{h}\tau \,, \tag{2.4}$$

where $\eta$ is the viscosity of the fluid.

The constitutive equation for an incompressible Newtonian fluid is

$$\tau = -pI + d \,, \tag{2.5}$$

where $p$ is the dimensionless hydrostatic pressure and $d$ is the rate of deformation tensor,

$$d = \tfrac{1}{2}(L + L^T) \,, \quad L = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \,. \tag{2.6}$$

For an incompressible fluid the conservation of mass yields the incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{2.7}$$

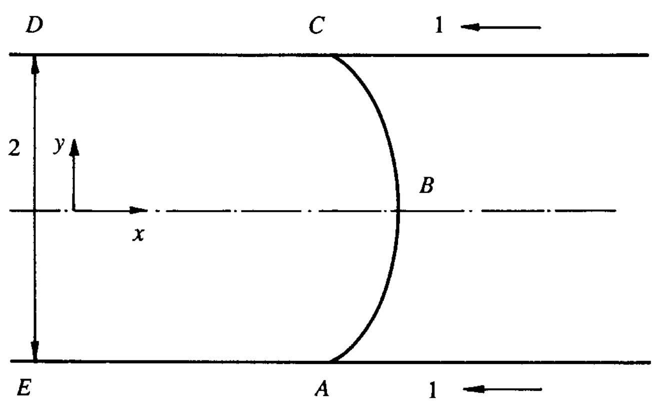


*Fig. 2.2.* The moving frame of reference.

This equation is satisfied by introducing a stream function $\psi = \psi(x, y)$,

$$\frac{\partial \psi}{\partial y} = u \,, \quad \frac{\partial \psi}{\partial x} = -v \,. \tag{2.8}$$

The stresses must satisfy the conservation of momentum. When body forces are absent and the accelerations can be neglected, we have for two-dimensional flow

$$t_{xx,x} + t_{xy,y} = 0 \,, \quad t_{xy,x} + t_{yy,y} = 0 \quad \left( {}_{,x} = \frac{\partial}{\partial x} \right) . \tag{2.9}$$

These equations are satisfied by introducing the Airy stress function $\phi = \phi(x, y)$,

$$t_{xx} = -\phi_{yy} \,, \quad t_{xy} = \phi_{xy} \,, \quad t_{yy} = -\phi_{xx} \,. \tag{2.10}$$

Furthermore, the pressure $p$ is related to $\phi$ by

$$p = -\tfrac{1}{2}(t_{xx} + t_{yy}) = \tfrac{1}{2}\Delta\phi = \tfrac{1}{2}(\phi_{xx} + \phi_{yy}) \,. \tag{2.11}$$

Because of the analogy between the theory of plane linear elasticity and the two-dimensional flow problem a description in complex functions as shown by Muskhelishvili [9, Ch. 5] can be used. Several problems for plane strain and generalized plane stress are treated by England [3, sec. 2.5]. An application to viscous fluid flow is given by Jacob [6, pp. 316–320]. Following these references we introduce the complex variables

$$z = x + \mathrm{i}y \quad \text{and} \quad \bar{z} = x - \mathrm{i}y \,. \tag{2.12}$$

All equations, including the constitutive relations (2.5), are satisfied by the introduction of two complex functions $\Omega(z)$ and $\omega(z)$ which are analytic in the domain $G_z$ occupied by the fluid. The general solution of the flow problem is then given by

$$\begin{aligned} &\phi + \mathrm{i}\psi = \bar{z}\Omega(z) + \omega(z) \,, \\ &w = u + \mathrm{i}v = z\overline{\Omega'(z)} + \overline{\omega'(z)} - \Omega(z) \,, \\ &t_{xx} + t_{yy} = -2[\Omega'(z) + \overline{\Omega'(z)}] \,, \\ &t_{xx} - t_{yy} + 2\mathrm{i}t_{xy} = 2[z\overline{\Omega''(z)} + \overline{\omega''(z)}] \,. \end{aligned} \tag{2.13}$$

The prime $'$ indicates differentiation with respect to the complex argument. The resulting force over an arc $PQ$ can also be expressed in the functions $\Omega(z)$ and $\omega(z)$, see [3, sec. 2.7], [9, sec. 33]. Along the arc we have a normal vector $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$, a tangent vector $\mathbf{s} = s_x\mathbf{e}_x + s_y\mathbf{e}_y = -n_y\mathbf{e}_x + n_x\mathbf{e}_y$, and the derivatives $\mathrm{d}z/\mathrm{d}s = s_x + \mathrm{i}s_y = \mathrm{i}(n_x + \mathrm{i}n_y)$ and $\mathrm{d}\bar{z}/\mathrm{d}s = s_x - \mathrm{i}s_y = -\mathrm{i}(n_x - \mathrm{i}n_y)$. For the normal and shear stresses $t_n$ and $t_s$ along the arc $PQ$ the following holds

$$\begin{aligned} t_n + \mathrm{i}t_s &= (\mathbf{n}, \tau\mathbf{n}) + \mathrm{i}(\mathbf{s}, \tau\mathbf{n}) = \\ &= t_{xx}n_x^2 + 2t_{xy}n_xn_y + t_{yy}n_y^2 + \mathrm{i}[t_{xx}n_xs_x + t_{xy}(n_xs_y + n_ys_x) + t_{yy}n_ys_y] = \\ &= \tfrac{1}{2}(t_{xx} + t_{yy}) + \tfrac{1}{2}(t_{xx} - t_{yy} + 2\mathrm{i}t_{xy})(n_x - \mathrm{i}n_y)^2 \,. \end{aligned}$$

Using (2.13) we derive

$$-(t_n + \mathrm{i}t_s)\,\frac{\mathrm{d}z}{\mathrm{d}s} = [\Omega'(z) + \overline{\Omega'(z)}]\,\frac{\mathrm{d}z}{\mathrm{d}s} + [z\overline{\Omega''(z)} + \overline{\omega''(z)}]\,\frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\,[\Omega(z) + z\overline{\Omega'(z)} + \overline{\omega'(z)}]\,.$$

Integrating over the arc $PQ$ we obtain the following expression for the resulting force:

$$K = -\int_P^Q (t_n + \mathrm{i}t_s)\,\mathrm{d}z = [z\overline{\Omega'(z)} + \overline{\omega'(z)} + \Omega(z)]_P^Q\,. \tag{2.14}$$

The functions $\Omega(z)$ and $\omega(z)$ are completely determined by the boundary conditions. Along every part of the boundary two conditions are necessary. Along the fluid surface, which is a free streamline of unknown shape, even three conditions are required.

It is assumed that there is complete adherence between the fluid and the straight walls AE and CD. This means $\mathbf{V} = \mathbf{0}$ there, or

$$u = -1\,, \quad v = 0\,, \quad y = \pm 1\,. \tag{2.15}$$

The environmental pressure is a constant denoted by $p_0$. Consequently, the normal stress $t_n$ and the shear stress $t_s$ must satisfy the following conditions along the free boundary ABC:

$$t_n = -p_0\,, \quad t_s = 0\,. \tag{2.16}$$

These conditions are substituted into the expression (2.14) for the resulting force over an arc $PQ$. Taking $P$ fixed and $Q$ in $z \in$ ABC we find

$$K = z\overline{\Omega'(z)} + \overline{\omega'(z)} + \Omega(z) = p_0 z + p_1\,, \tag{2.17}$$

where $p_1 \in \mathbf{C}$ is an integration constant.

The third condition along the fluid surface ABC follows from the assumption that its shape does not alter. This means that the normal velocity relative to the moving frame must vanish,

$$(\mathbf{v}, \mathbf{n}) = 0\,, \tag{2.18}$$

where $\mathbf{n}$ denotes the outer normal to the surface ABC.

For a complete determination of the mathematical problem conditions at infinity ($x \to -\infty$) are required. At large distance of the flow front the flow will resemble the fully developed flow, the so-called Poiseuille flow, which will be denoted by an index 0. This type of flow occurs when the space between the two parallel plates would be completely filled with fluid. The limiting value of the velocity must be

$$u \to u_0 = \tfrac{1}{2} - \tfrac{3}{2}y^2\,, \quad v \to v_0 = 0\,, \quad (x \to -\infty)\,. \tag{2.19}$$

The procedure of solution is to calculate the functions $\Omega(z)$ and $\omega(z)$ subject to the boundary conditions (2.15), (2.17), and (2.19), and subsequently to determine the free boundary with condition (2.18).

Because of the inhomogeneity of the conditions (2.15) and (2.19) for the velocity it proves to be convenient to subtract the Poiseuille flow. We write

$$u = u_0 + u_1 , \quad v = v_0 + v_1 , \quad \text{and} \quad w_1 = u_1 + \mathrm{i}v_1 . \tag{2.20}$$

The velocities $u_0$ and $v_0$ are given by (2.19), while $u_1$ and $v_1$ are the new unknown functions. We replace $\Omega(z)$ and $\omega(z)$ by $\Omega_0(z) + \Omega_1(z)$ and $\omega_0(z) + \omega_1(z)$, respectively, with

$$\Omega_0(z) = -\tfrac{1}{4}(1 + \tfrac{3}{2}z^2) + \tfrac{1}{2}p_2 z , \quad \omega_0(z) = \tfrac{1}{4}z(1 + \tfrac{1}{2}z^2) , \tag{2.21}$$

representing the Poiseuille flow and with $\Omega_1(z)$ and $\omega_1(z)$ the new unknown functions. The constant $p_2 \in \mathbf{R}$ represents a uniform pressure and is still free to be chosen.

From (2.13) and (2.21) we find

$$\begin{aligned} w_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} - \Omega_1(z) , \\ t_{xx} + t_{yy} &= -2[\Omega_1'(z) + \overline{\Omega_1'(z)}] + \tfrac{3}{2}(z + \bar{z}) - 2p_2 , \\ t_{xx} - t_{yy} + 2\mathrm{i}t_{xy} &= 2[z\overline{\Omega_1''(z)} + \overline{\omega_1''(z)}] - \tfrac{3}{2}(z - \bar{z}) , \end{aligned} \tag{2.22}$$

and from (2.14) for the resulting force along an arc $PQ$,

$$\begin{aligned} K &= -\int_P^Q (t_n + \mathrm{i}t_s)\,\mathrm{d}z = \\ &= [z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} + \Omega_1(z) - \tfrac{3}{8}(z^2 + 2z\bar{z} - \bar{z}^2) + p_2 z]_P^Q . \end{aligned} \tag{2.23}$$

The boundary conditions (2.15), (2.17), and (2.19) transform into

$$\begin{aligned} w_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} - \Omega_1(z) = 0 , \quad y = \pm 1 , \\ w_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} - \Omega_1(z) \to 0 , \quad (x \to -\infty) , \\ K_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} + \Omega_1(z) = \tfrac{3}{8}(z^2 + 2z\bar{z} - \bar{z}^2) + (p_0 - p_2)z + p_1 , \quad z \in \mathrm{ABC} . \end{aligned} \tag{2.24}$$

Choosing $p_2 = p_0$ and omitting the irrelevant constant $p_1$, we have

$$K_1 = \tfrac{3}{8}(z^2 + 2z\bar{z} - \bar{z}^2) , \quad z \in \mathrm{ABC} . \tag{2.25}$$

## 3. The conformal mapping

For the solution of the problem described in the preceding section conformal-mapping techniques will be used. The domain $G_z$ occupied by the fluid is transformed into the interior of the unit circle, $G_\zeta^+ := \{\zeta \in \mathbf{C} \mid |\zeta| < 1\}$ (see Figs. 3.1 and 3.2).

The mapping function will be denoted by

$$z = \mathrm{m}(\zeta) . \tag{3.1}$$

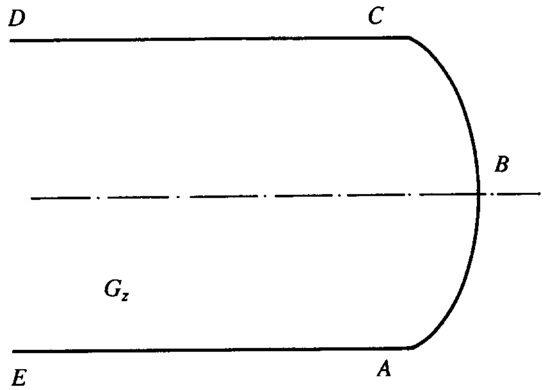


*Fig. 3.1.* The domain $G_z$.

This transformation is conformal, implying that the function $m(\zeta)$ is an analytic and univalent function for $\zeta \in G_\zeta^+$. Further, it is assumed that the mapping function is continuous on $\overline{G_\zeta^+}$, except in the point $\zeta = -1$.

The conformal mapping of $G_\zeta^+$ onto $G_z$ exists and is uniquely determined by the choice of the points B, C, and D on the unit circle. This is a result from the Riemann mapping theorem. We shall now consider the limit for $\zeta$ tending to a point $\xi$ on the boundary of $G_\zeta^+$, $|\xi| = 1$, $\xi \neq -1$. The corresponding point $z = m(\zeta)$ will tend to a point of the boundary of $G_z$. The limiting value of $z$ is

$$m^+(\xi) = \lim_{\zeta \to \xi, \zeta \in G_\zeta^+} m(\zeta) = x(\xi) + iy(\xi)\,. \tag{3.2}$$

The complex parameter $\xi$ is related to the arclength $s$ along the boundary of $G_z$, $\xi = \xi(s)$ say. Then, the following relations for the tangential vector $\mathbf{s} = s_x\mathbf{e}_x + s_y\mathbf{e}_y$ and the normal vector $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$ along the boundary of $G_z$ exist

$$\begin{aligned} s_x + is_y &= \frac{dx}{ds} + i\,\frac{dy}{ds} = \frac{dz}{ds} = m'^+(\xi)\,\frac{d\xi}{ds}\,, \\ n_x + in_y &= s_y - is_x = -i\,m'^+(\xi)\,\frac{d\xi}{ds}\,. \end{aligned} \tag{3.3}$$

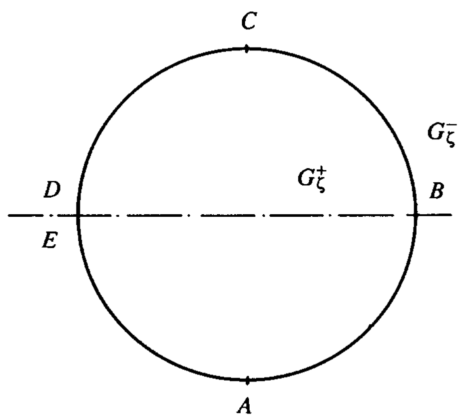


*Fig. 3.2.* The $\zeta$-plane.

We now eliminate the derivative $\mathrm{d}\xi/\mathrm{d}s$. Since $\xi$ is on the unit circle, we have

$$\xi\bar{\xi} = 1 . \tag{3.4}$$

Differentiating (3.4) with respect to the arclength $s$, we obtain

$$\bar{\xi}\,\frac{\mathrm{d}\xi}{\mathrm{d}s} + \xi\,\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}s} = 0 . \tag{3.5}$$

Combining (3.3) and (3.5) we find a relation between the components of the normal vector and the derivative of the conformal mapping function, reading

$$\frac{n_x + \mathrm{i}n_y}{n_y - \mathrm{i}n_y} = \frac{-\mathrm{i}\,\mathrm{m}'^{+}(\xi)\,\dfrac{\mathrm{d}\xi}{\mathrm{d}s}}{\mathrm{i}\,\overline{\mathrm{m}'^{+}(\xi)}\,\dfrac{\mathrm{d}\bar{\xi}}{\mathrm{d}s}} = \frac{\xi\mathrm{m}'^{+}(\xi)}{\overline{\xi\mathrm{m}'^{+}(\xi)}} . \tag{3.6}$$

The relation is not valid in the point $\xi = -1$, because this point corresponds to infinity in the complex $z$-plane. This point, $\zeta = -1$, is a singular point of the conformal mapping. The character of this singularity is logarithmic. This implies, that the introduction of branch cuts from $\zeta = -1$ to $\zeta = \infty$ and along the arc ABC is necessary when the function $\mathrm{m}(\zeta)$ is continuated to the exterior of the unit circle $G_\zeta^- := \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}$. Analogous to the theory of linear elasticity [3, Ch. 5], [9, Ch. 15, 21], we shall approximate the conformal mapping function $\mathrm{m}(\zeta)$ by a polynomial. This technique has been applied successfully to several problems with given boundaries. Thereby, difficulties caused by branch cuts are avoided. The basis of this approximation is the Taylor expansion of the mapping function,

$$\mathrm{m}(\zeta) = \sum_{k=0}^{\infty} \mu_k \zeta^k , \quad |\zeta| < 1 . \tag{3.7}$$

The radius of convergence of this series is 1. The series is still converging for $|\zeta| = 1$ except in $\zeta = -1$. For reasons of symmetry the coefficients $\mu_k$ are real. Truncating the series after $N + 1$ terms, we have a polynomial of degree $N$,

$$\mathrm{m}_N(\zeta) = \sum_{k=0}^{N} \mu_k \zeta^k . \tag{3.8}$$

Since the shape of the free boundary ABC is to be determined, the conformal mapping and its coefficients $\mu_k$ are unknown. In the following sections the coefficients $\mu_k$, $0 \leqslant k \leqslant N$, given $N$, will be calculated such that all conditions are satisfied as well as possible. A specification of these conditions is given in the last part of Section 5. After the coefficients are determined, an approximation of the shape of the fluid surface is given by

$$z = \mathrm{m}_N(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=0}^{N} \mu_k\,\mathrm{e}^{\mathrm{i}k\theta} , \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi . \tag{3.9}$$

We assume, that the polynomial $\mathrm{m}_N(\zeta)$ produces a good approximation of $\mathrm{m}(\zeta)$ in the neighbourhood of the free boundary ABC, for instance in the right half of the interior of the unit circle $D := \{\zeta \in \mathbb{C} \mid |\zeta| < 1, \operatorname{Re} \zeta > 0\}$.

The function $m_N(\zeta)$ can be defined not only in the domain $G_\zeta^+$ but also in $G_\zeta^-$. To this end, the same functional prescription (3.8) is applied. The continuated function $m_N(\zeta)$ is analytic for all $\zeta \in \mathbf{C}$ and branch cuts do not occur. A physical interpretation of $m_N(\zeta)$ in the region $G_\zeta^-$ is not possible, because this domain does not correspond with any part of the flow region.

It has not yet been shown that the function $m_N(\zeta)$ is a conformal mapping, i.e. analytic and univalent for $\zeta \in G_\zeta^+$. Since $m_N(\zeta)$ is a polynomial, it is clearly an analytic function. The univalence of a conformal mapping is equivalent to the condition that its derivative doesn't vanish. It is therefore assumed, that the polynomial $m_N(\zeta)$ satisfies this condition in the domain $G_\zeta^+$. Once the coefficients $\mu_k$ have been determined, this assumption must be verified. Since our main interest lies in the free boundary, it is sufficient that $m_N'(\zeta)$ doesn't vanish in the region $D \subset G_\zeta^+$.

We expect that, with increasing $N$, the function $m_N(\zeta)$ will produce a better approximation of the conformal mapping $m(\zeta)$. The domain onto which the interior of the unit circle $G_\zeta^+$ is mapped by $m_N(\zeta)$ will resemble the domain $G_z$ onto which $G_\zeta^+$ is mapped by $m(\zeta)$, for large $N$, see [1, Sec. 104] and [5, Ch. 1, Sec. 5, Th. 1-2]. In the following sections an approximate solution of the flow problem is derived by means of an analytic continuation of complex functions, in analogy with the Muskhelishvili method in linear elasticity.

## 4. Transformation of the problem

The flow of the fluid is described by the functions $\Omega_1(z)$ and $\omega_1(z)$. The complex velocity and the stresses are expressed in these two functions by the relations (2.22). The functions $\Omega_1(z)$ and $\omega_1(z)$ are completely determined by the boundary conditions (2.24) and (2.25).

The conformal mapping transforms the domain $G_z$ occupied by the fluid into the unit circle of the complex $\zeta$-plane. Consequently, the velocity and the stresses must be expressed in the complex variable $\zeta$. To this end, we write

$$\Omega_1(z) = \Omega_1(m(\zeta)) \equiv \tilde{\Omega}_1(\zeta)\,, \quad \omega_1(z) = \omega_1(m(\zeta)) \equiv \tilde{\omega}_1(\zeta)\,. \tag{4.1}$$

The derivatives are given by

$$\Omega_1'(z) = \tilde{\Omega}_1'(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{\tilde{\Omega}_1'(\zeta)}{m'(\zeta)}\,, \quad \omega_1'(z) = \tilde{\omega}_1'(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{\tilde{\omega}_1'(\zeta)}{m'(\zeta)}\,. \tag{4.2}$$

If the conformal mapping $m(\zeta)$ is approximated by a polynomial $m_N(\zeta)$, the function $\tilde{\Omega}_1(\zeta)$ and $\tilde{\omega}_1(\zeta)$ must be replaced by $\Omega_N(\zeta)$ and $\omega_N(\zeta)$ respectively. It is emphasized that these two functions are not necessarily polynomials.

The complex velocity and the stresses can now be expressed in the variable $\zeta$ and in the complex functions $m_N(\zeta)$, $\Omega_N(\zeta)$, and $\omega_N(\zeta)$. From (2.22), we find for the approximations

$$\begin{aligned} w_{N1} &= \frac{m_N(\zeta)\overline{\Omega_N'(\zeta)} + \overline{\omega_N'(\zeta)}}{\overline{m_N'(\zeta)}} - \Omega_N(\zeta)\,, \\ t_{xx} + t_{yy} &= -2\left[\frac{\Omega_N'(\zeta)}{m_N'(\zeta)} + \frac{\overline{\Omega_N'(\zeta)}}{\overline{m_N'(\zeta)}}\right] + \tfrac{3}{4}(m_N(\zeta) + \overline{m_N(\zeta)}) - 2p_0\,, \\ t_{xx} - t_{yy} + 2\mathrm{i}\,t_{xy} &= \frac{2}{\overline{m_N'(\zeta)}}\left[m_N(\zeta)\,\overline{\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\Omega_N'(\zeta)}{m_N'(\zeta)}\right)} + \overline{\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\omega_N'(\zeta)}{m_N'(\zeta)}\right)}\right] - \tfrac{3}{4}(m_N(\zeta) - \overline{m_N(\zeta)})\,. \end{aligned} \tag{4.3}$$

The limiting values of the functions $\Omega_N(\zeta)$ and $\omega_N(\zeta)$ for $\zeta$ tending to a point $\xi$ on the boundary of $G_\zeta^+$, $|\xi|=1$, are denoted by

$$\Omega_N^+(\xi) = \lim_{\zeta\to\xi,\zeta\in G_\zeta^+} \Omega_N(\zeta)\,, \quad \omega_N^+(\xi) = \lim_{\zeta\to\xi,\zeta\in G_\zeta^+} \omega_N(\zeta)\,. \tag{4.4}$$

The boundary conditions for $\Omega_N(\zeta)$ and $\omega_N(\zeta)$ on the unit circle follow from (2.24) and (2.25),

$$\frac{\mathrm{m}_N^+(\xi)\overline{\Omega_N'^+(\xi)} + \overline{\omega_N'^+(\xi)}}{\overline{\mathrm{m}_N'^+(\xi)}} - \Omega_N^+(\xi) = 0\,, \qquad \xi \in \mathrm{CDEA}\,, \tag{4.5}$$

$$\frac{\mathrm{m}_N^+(\xi)\overline{\Omega_N'^+(\xi)} + \overline{\omega_N'^+(\xi)}}{\overline{\mathrm{m}_N'^+(\xi)}} + \Omega_N^+(\xi) = g_N(\xi)\,, \quad \xi \in \mathrm{ABC}\,, \tag{4.6}$$

with $g_N(\xi)$ defined by

$$g_N(\xi) := \tfrac{3}{8}([\mathrm{m}_N^+(\xi)]^2 + 2\mathrm{m}_N^+(\xi)\overline{\mathrm{m}_N^+(\xi)} - [\overline{\mathrm{m}_N^+(\xi)}]^2)\,, \quad \xi \in \mathrm{ABC}\,. \tag{4.7}$$

The condition (2.18) for the determination of the free boundary can be written as (with $w$ replaced by $w_N$)

$$\mathrm{Re}[w_N(n_x - \mathrm{i}n_y)] = 0\,,$$

which, with use of (3.6), leads to

$$\mathrm{Re}[w_N\xi\overline{\mathrm{m}_N'^+(\xi)}] = 0\,, \quad \xi \in \mathrm{ABC}\,, \tag{4.8}$$

where, in accordance with (2.19) and (2.20),

$$w_N = \tfrac{1}{2} + \tfrac{3}{8}([\mathrm{m}_N^+(\xi)]^2 - 2\mathrm{m}_N^+(\xi)\overline{\mathrm{m}_N^+(\xi)} + [\overline{\mathrm{m}_N^+(\xi)}]^2) + w_{N1}\,. \tag{4.9}$$

The functions $\Omega_N(\zeta)$ and $\omega_N(\zeta)$ are to be determined from the boundary conditions (4.5) and (4.6). These equations can be solved by a continuation of $\Omega_N(\zeta)$ to the exterior of the unit circle, $G_\zeta^-$. This continuation is denoted by $\Psi_N(\zeta)$ and is defined by

$$\Psi_N(\zeta) := \begin{cases} \Omega_N(\zeta)\,, & \zeta \in G_\zeta^+\,, \\ \dfrac{\mathrm{m}_N(\zeta)\overline{\Omega_N'(1/\bar\zeta)} + \overline{\omega_N'(1/\bar\zeta)}}{\overline{\mathrm{m}_N'(1/\bar\zeta)}}\,, & \zeta \in G_\zeta^-\,. \end{cases} \tag{4.10}$$

The function $\Psi_N(\zeta)$ is analytic for $\zeta \in G_\zeta^+$ and for $\zeta \in G_\zeta^-$. Once $\Psi_N(\zeta)$ is known, the functions $\Omega_N(\zeta)$ and $\omega_N(\zeta)$ follow from definition (4.10). For $\zeta \in G_\zeta^+$ we have

$$\Omega_N(\zeta) = \Psi_N(\zeta)\,, \tag{4.11}$$

$$\omega_N'(\zeta) = \mathrm{m}_N'(\zeta)\overline{\Psi_N(1/\bar\zeta)} - \overline{\mathrm{m}_N(1/\bar\zeta)}\Psi_N'(\zeta)\,. \tag{4.12}$$

Since $m_N(\zeta)$ is a polynomial of degree $N$, the functions $\overline{m_N(1/\bar{\zeta})}$ and $\overline{\Psi_N(1/\bar{\zeta})}$ have poles of order $N$ in the origin $\zeta = 0$. However, $\omega_N'(\zeta)$ is an analytic function for $\zeta \in G_\zeta^+$ including the origin and hence, the right-hand side of equation (4.12) must remain bounded for $\zeta \to 0$. This condition is known as the holomorphy condition (see [3, Sec. 5.4]).

Substitution of (4.10) into the boundary conditions (4.5) and (4.6) yields

$$\Psi_N^-(\xi) - \Psi_N^+(\xi) = 0\,, \qquad \xi \in \mathrm{CDEA}\,, \tag{4.13}$$

$$\Psi_N^-(\xi) + \Psi_N^+(\xi) = g_N(\xi)\,, \qquad \xi \in \mathrm{ABC}\,. \tag{4.14}$$

Expressing the complex velocity $w_{N1}$ in the function $\Psi_N(\zeta)$, we find

$$w_{N1} = \Psi_N(1/\bar{\zeta}) - \Psi_N(\zeta) + \frac{m_N(\zeta) - m_N(1/\bar{\zeta})}{\overline{m_N'(\zeta)}}\,\overline{\Psi_N'(\zeta)}\,. \tag{4.15}$$

Since the velocity must remain finite near the points A and C, we have the following condition

$$\Psi_N(\zeta) = O(1)\,, \quad (\zeta \to \pm \mathrm{i})\,. \tag{4.16}$$

From (4.13), we conclude that $\Psi_N(\zeta)$ is continuous over the arc CDEA and, therefore, is an analytic function for $\zeta \in \mathbb{C}\backslash\mathrm{ABC}$. Along the cut ABC the jump condition (4.14) holds and near the endpoints A and C condition (4.16) must be satisfied. This problem for the function $\Psi_N(\zeta)$ is called a Hilbert problem. Its solution is derived in the next section.

## 5. The solution of the Hilbert problem

The theory for the solution of Hilbert problems has extensively been treated by Muskhelishvili [8] and [9, Ch. 18]. A summary of this theory is given by England [3, Ch. 1]. The equation (4.14) along the arc ABC and the condition (4.16) near the endpoints A and C produce the Hilbert problem for the function $\Psi_N(\zeta)$. The general solution is given by

$$\Psi_N(\zeta) = X(\zeta)G_N(\zeta) + X(\zeta)F_N(\zeta)\,, \quad \zeta \in \mathbb{C}\backslash\mathrm{ABC}\,, \tag{5.1}$$

where $G_N(\zeta)$ is defined by

$$G_N(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{ABC}} \frac{g_N(\xi)}{X^+(\xi)(\xi - \zeta)}\,\mathrm{d}\xi\,, \quad \zeta \in \mathbb{C}\backslash\mathrm{ABC}\,, \tag{5.2}$$

while the function $X(\zeta)$ is the characteristic Plemelj function defined by

$$X(\zeta) = (\zeta - \mathrm{i})^{1/2}(\zeta + \mathrm{i})^{1/2}\,, \qquad \zeta \in \mathbb{C}\backslash\mathrm{ABC}\,, \tag{5.3}$$

having a branch cut along the arc ABC. The function $F_N(\zeta)$ still has to be determined. The branch cuts for the roots in the function $X(\zeta)$ are chosen in the following way. The branch for $\zeta + \mathrm{i}$ is from $-\mathrm{i}$ to $-\mathrm{i}\infty$ along the imaginary axis, so $-\frac{1}{2}\pi \leq \arg(\zeta + \mathrm{i}) \leq \frac{3}{2}\pi$. The branch for $\zeta - \mathrm{i}$ is from $+\mathrm{i}$ to $-\mathrm{i}$ along the arc ABC and from $-\mathrm{i}$ to $-\mathrm{i}\infty$ along the imaginary axis.

This means that $X(0) = -1$ and

$$X(\zeta) = \zeta + O(1/\zeta), \quad (|\zeta| \to \infty). \tag{5.4}$$

As the function $X(\zeta)$ is continuous across the part of the imaginary axis from $-\mathrm{i}$ to $-\mathrm{i}\infty$, the branch cut reduces to the arc ABC and $X(\zeta)$ is analytic for $\zeta \in \mathbf{C}\backslash\mathrm{ABC}$. Along the arc ABC one has

$$X^+(\xi) + X^-(\xi) = 0, \quad \xi \in \mathrm{ABC}. \tag{5.5}$$

The function $G_N(\zeta)$ is analytic for $\zeta \in \mathbf{C}\backslash\mathrm{ABC}$ and

$$G_N(\zeta) = O(1/\zeta), \quad (|\zeta| \to \infty). \tag{5.6}$$

With use of the Plemelj formulae for Cauchy integrals (see [8, sec. 17], [9, sec. 68]) we derive

$$X^+(\xi)G_N^+(\xi) + X^-(\xi)G_N^-(\xi) = g_N(\xi), \quad \xi \in \mathrm{ABC}. \tag{5.7}$$

This means that $X(\zeta)G_N(\zeta)$ is a particular solution of equation (4.14). From an expansion of $G_N(\zeta)$ near the endpoints A and C we find (see [8, sec. 29], [9, sec. 110])

$$X(\zeta)G_N(\zeta) = O(1), \quad (\zeta \to \pm\mathrm{i}). \tag{5.8}$$

The function $G_N(\zeta)$ can be calculated explicitly in terms of the coefficients $\mu_k$, $0 \leqslant k \leqslant N$, by means of contour integration.

We proceed with the determination of the function $F_N(\zeta)$, which is proven to be analytic for $\zeta \in \mathbf{C}\backslash\mathrm{ABC}$ only. From (5.1) and (5.7), we conclude that $X(\zeta)F_N(\zeta)$ is a homogeneous solution of equation (4.14), i.e.

$$X^+(\xi)F_N^+(\xi) + X^-(\xi)F_N^-(\xi) = 0, \quad \xi \in \mathrm{ABC},$$

and with (5.5) it then follows that

$$F_N^+(\xi) = F_N^-(\xi), \quad \xi \in \mathrm{ABC}. \tag{5.9}$$

Hence, $F_N(\zeta)$ is analytic for $\zeta \in \mathbf{C}\backslash\{\mathrm{i}, -\mathrm{i}\}$.

Due to (4.16) and (5.8),

$$X(\zeta)F_N(\zeta) = O(1), \quad (\zeta \to \pm\mathrm{i}),$$

or

$$F_N(\zeta) = O((\zeta \mp \mathrm{i})^{-1/2}), (\zeta \to \pm\mathrm{i}), \tag{5.10}$$

and, thus, the singularities of $F_N(\zeta)$ in $\zeta = \pm\mathrm{i}$ are removable. So we conclude that $F_N(\zeta)$ is an entire function, i.e. analytic for all $\zeta \in \mathbf{C}$. Since, according to (4.10), $\Psi_N(\zeta) = O(\zeta^N)$, $(|\zeta| \to \infty)$, the function $F_N(\zeta)$ must be a polynomial of degree $N-1$,

$$F_N(\zeta) = \sum_{k=0}^{N-1} f_k \zeta^k, \quad \zeta \in \mathbf{C}. \tag{5.11}$$

For reasons of symmetry the coefficients $f_k$, $0 \leq k \leq N-1$, are real. These coefficients are determined by applying the holomorphy condition to (4.12), producing

$$\mathrm{m}_N'(\zeta)\overline{\Psi_N(1/\bar{\zeta})} - \overline{\mathrm{m}_N(1/\bar{\zeta})}\Psi_N'(\zeta) = O(1) , \quad (\zeta \to 0) . \tag{5.12}$$

This condition yields $N$ linear equations for the unknown $f_k$, $0 \leq k \leq N-1$.

The functions $F_N(\zeta)$ and $G_N(\zeta)$ and, hence, also $\Psi_N(\zeta)$ (see (5.1)) are now entirely expressed in the coefficients $\mu_k$, $0 \leq k \leq N$, of the conformal mapping $\mathrm{m}_N(\zeta)$. Relation (4.8) gives a criterion for the determination of these coefficients. Since the exact mapping function $\mathrm{m}(\zeta)$ is approximated by a polynomial of degree $N$, a discrete set of $N+1$ equations is required. First, we impose for geometrical reasons that the point C, $\zeta = \mathrm{i}$, is mapped onto $z = \mathrm{i}$ in the domain $G_z$, i.e.

$$\mathrm{m}_N(\mathrm{i}) = \mathrm{i} . \tag{5.13}$$

The point A, $\zeta = -\mathrm{i}$, is then mapped onto $z = -\mathrm{i}$. The real and imaginary parts of (5.13) yield two equations for the coefficients $\mu_k$, $0 \leq k \leq N$. The other $N-1$ equations are derived by demanding that the normal velocity vanishes in $N-1$ points of the arc ABC. Because of the symmetry of the problem, we restrict ourselves to the arc BC. On this arc we choose the points

$$\xi_k := \mathrm{e}^{\mathrm{i}\theta_k} , \quad \theta_k := \frac{\pi k}{2N} , \quad 1 \leq k \leq N-1 . \tag{5.14}$$

The coefficients $\mu_k$, $0 \leq k \leq N$, are now determined by (5.13) and by the conditions

$$\mathrm{Re}[w_N \overline{\xi_k \mathrm{m}_N'^{+}(\xi_k)}] = 0 , \quad 1 \leq k \leq N-1 . \tag{5.15}$$

These equations are solved by a numerical procedure for the solution of systems of non-linear equations. An approximation of the shape of the free boundary ABC is then given by the relation (3.9). The results are presented in the final section.

## 6. Results and conclusions

The approximation of the exact mapping function $\mathrm{m}(\zeta)$ by a polynomial $\mathrm{m}_N(\zeta)$ of degree $N$, see (3.8), and subsequent calculation of the function $\Psi_N(\zeta)$, as described in the preceding section, has been carried out for $N = 3$, 4, 5, and 6. The results for the coefficients $\mu_k$, $0 \leq k \leq N$, are listed in Table 6.1. Estimates of the errors in the numerical solution of the equations (5.13) and (5.15) for the coefficients $\mu_k$ are also computed. They are in the order of $10^{-5}$, if $N = 3$, 4, 5. Because of the non-linearity of the equations (5.15) for the coefficients $\mu_k$, the numerical procedure for the solution of these equations is slowly converging when $N$ becomes too large. Nevertheless, the coefficients $\mu_k$ can be calculated for $N = 6$ with an error of $10^{-3}$ at most, i.e. less than 1%. For higher values of $N$ convergence is too slow to produce more than 3-digit accuracy which means an error of 1% at most. On the other hand, no further improvement of the outcome is observed when $N = 7$ or 8. So the results for $N = 4$, 5, 6 are satisfactory.

As has been said in Section 3, it must be verified that the polynomial $\mathrm{m}_N(\zeta)$ is a conformal mapping, i.e., that the derivative $\mathrm{m}_N'(\zeta)$ doesn't vanish in the domain $G_\zeta^+$. The total number

*Table 6.1.* The coefficients $\mu_k$ for several values of $N$

| $k$ | $N=3$ | $N=4$ | $N=5$ | $N=6$ |
|---|---|---|---|---|
| 0 | −0.04287 | −0.01699 | 0.01997 | 0.125 |
| 1 | 0.98349 | 0.95126 | 0.88278 | 0.689 |
| 2 | −0.04287 | −0.00065 | 0.07560 | 0.307 |
| 3 | −0.01651 | −0.04874 | −0.12904 | −0.368 |
| 4 | | 0.01634 | 0.05563 | 0.189 |
| 5 | | | −0.01182 | −0.057 |
| 6 | | | | 0.008 |

of zeroes of $m_N'(\zeta)$ inside a contour $\Gamma$ is given by the integral

$$I = \frac{1}{2\pi i} \int_\Gamma \frac{m_N''(\zeta)}{m_N'(\zeta)} \, d\zeta \, . \tag{6.1}$$

Calculation of the integral $I$ with $\Gamma$ being the unit circle proves that the function $m_N(\zeta)$ is conformal for $N = 3$, 4, and 5. In the case $N = 6$ the function $m_N(\zeta)$ is conformal in the right half $D$ of the unit circle. This can be shown by evaluating the integral $I$ with $\Gamma$ being the boundary of the region $D$.

The shape of the free boundary is calculated from relation (3.9). It appears that the curves for $N = 5$ and $N = 6$ are lying between those for $N = 3$ and $N = 4$. So we can say that the curves are converging to the exact free boundary. The difference between the successive approximations of the free boundary is about 2–3% of the semi-distance of the two plates. Since there exists no visual distinction between all the approximations, we confine ourselves to showing the fluid surface in Figure 6.1 in the case for $N = 5$ only.

By the equation (4.15) the velocity $w$ is related to the mapping function $m_N(\zeta)$ and the function $\Psi_N(\zeta)$. These two functions can be expressed in the coefficients $\mu_k$ and therefore the velocity $w$ is known when the coefficients are calculated. In Fig. 6.1 streamlines are drawn in the region behind the flow front. When approaching the front the streamlines diverge. This typical behaviour is called the fountain effect, as indicated before.

We conclude that the shape of the fluid surface can be calculated by means of a polynomial approximation of the conformal mapping function which maps the flow region onto the unit circle. Taking degree $N = 4$, 5, 6 for the polynomial already produces good results for the free boundary. These results only differ by 2–3% from those obtained by Dierieck [2] and Mavridis, Hrymak and Vlachopoulos [7], who used an iterative scheme for the determination of the free boundary. Numerical efforts involved in such an iteration are avoided in this paper. Calculations are restricted to the solution of the equations (5.13) and (5.15) for the coefficients of the conformal mapping function. When these coefficients are computed, the velocity of the fluid can be calculated in the neighbourhood of the free surface with the relation (4.15). The characteristical fountain effect of the diverging streamlines is demonstrated in Fig. 6.1.

The method of complex functions and conformal mapping, which is up to now mostly utilized to solve problems in mathematical physics with prescribed boundaries, appears to be just as well useful for solving free-boundary problems in viscous fluid flow. This technique has also been applied to the die-swell problem for the extrusion of a fluid from a capillary. This research has been done in cooperation with A.J.M. Sipers. The results are presented in a companion paper (*this Journal* 24 (1990) 167–178).

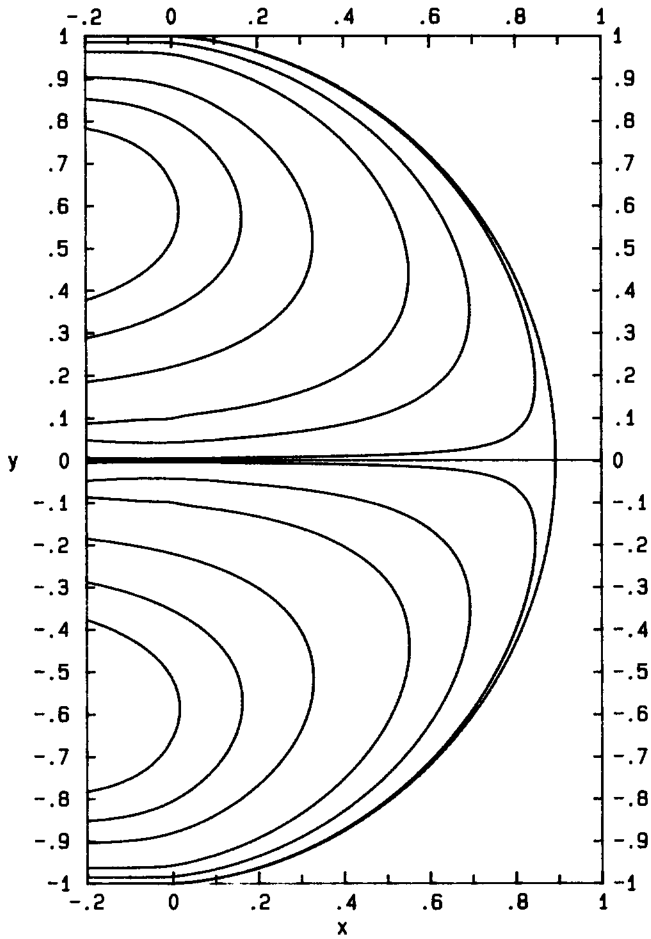


*Fig. 6.1.* The free boundary.

**Acknowledgements**

We wish to thank A.A.F. van de Ven for his guidance and many helpful discussions. We also want to express our gratitude to A.J.M. Sipers for his work and ideas on our research of viscous fluid flow with free boundaries.

## References

1. Carathéodory, C., *Conformal representation*, Cambridge: University Press (1958).
2. Dierieck, C., *Stream function formulation of Stokes problems with stress boundary conditions*, Brussels: Philips Research Laboratory, Report R479 (1984).
3. England, A.H., *Complex variable methods in elasticity*, London: Wiley-Interscience (1971).
4. Garabedian, P.R., Free boundary flows of a viscous liquid, *Comm. on Pure and Appl. Math.* 19 (1966) 421–434.
5. Goluzin, G.M., *Geometric theory of functions of a complex variable*, Providence RI: Am. Math. Soc. (1969).
6. Jacob, C., *Introduction mathématique à la mécanique des fluides*, Paris: Gauthier-Villars (1959).
7. Mavridis, H., A.N. Hrymak and J. Vlachopoulos, Finite element simulation of fountain flow in injection moulding, *Polymer Engng. and Sci.* 26 (1986) 449–454.
8. Muskhelishvili, N.I., *Singular integral equations*, Groningen: Noordhoff (1953).
9. Muskhelishvili, N.I., *Some basic problems of the mathematical theory of elasticity*, Groningen: Noordhoff (1953).